



Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at <http://about.jstor.org/participate-jstor/individuals/early-journal-content>.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

GREEK METHODS OF SOLVING QUADRATIC EQUATIONS.

By WALTER C. EELLS, The University of Chicago.

The following Bibliography includes the chief references consulted in the preparation of this paper.

CANTOR, M.:—*Vorlesungen über Geschichte der Mathematik*. Band 1. Zweite Auflage. Leipzig, 1894.

NESSELMANN, G. H. P.:—*Die Algebra der Griechen*. Berlin, 1842.

GOW, J.:—*A Short History of Greek Mathematics*. Cambridge, 1884.

ALLMAN, G. J.:—*Greek Geometry from Thales to Euclid*. Dublin, 1889.

TANNERY, P.:—*Diophanti Alexandrini; Opera omnia cum Graecis Commentariis*. 2 Vols. Lipsiae, 1893. (Complete Greek text of Diophantus with Latin translation).

WERTHEIM, G.:—*Die Arithmetik und die Schrift über Polygonalzahlen des Diophantus von Alexandria*. Leipzig, 1890. (Translation in German of the Greek text of Diophantus).

HANKEL, H.:—*Zur Geschichte der Mathematik in Alterthum und Mittelalter*. Leipzig, 1874.

ZEUTHEN, H. G.:—*Histoire des Mathématiques*. Paris, 1902.

HEIBERG, I. L.:—*Euclidis Elementa*. Lipsiae, 1883. 2 Vols. (Greek text with Latin translation).

HEIBERG, I. L.:—*Archimedis, Opera Omnia*. Vol. I. Lipsiae, 1880. (Greek text with Latin translation).

HEATH, T. L.:—*The Thirteen Books of Euclid's Elements*. Cambridge, 1908. 3 Vols. (Translation in English of the Greek text of Euclid).

HEATH, T. L.:—*The Works of Archimedes*. Cambridge, 1897.

HEATH, T. L.:—*Diophantus of Alexandria, a Study in the History of Greek Algebra*. Cambridge, 1885.

HEATH, T. L.:—*Apollonius, Conic Sections*. Cambridge, 1896.

LA GRANGE, J. L.:—*Oeuvres*. Paris, 1877. *Leçons Élémentaires sur les Mathématiques*. Vol. 7, p. 180.

BALL, W. W. R.:—*A Short History of Mathematics*. London, 1893.

CAJORI, F.:—*History of Mathematics*. New York, 1894.

FINK, K.:—*History of Mathematics*. (Beman and Smith translation) Chicago, 1900.

File of the *Bibliotheca Mathematica*, G. Enestrom, Leipzig.

Principal Greek mathematicians referred to in this paper with the approximate dates at which they flourished:—PYTHAGORAS, 530 B. C.; EUCLID, 290 B. C.; ARCHIMEDES, 250 B. C.; APOLLONIUS, 230 B. C.; HIPPARCHUS, 150 B. C.; HERON (HERO), 120 B. C.; DIOPHANTUS, 275 A. D.; THEON, 380 A. D.; PROCLUS, 450 A. D.

INTRODUCTION.

Before entering upon a discussion of the methods employed by the ancient Greeks for solving Quadratic Equations, a brief summary should be made of the mathematical knowledge which they possessed in historic times. Various forms of reckoning, including finger-reckoning, pebble-reckoning, and some use of the sand-board ($\alpha\beta\alpha'\varepsilon$) especially characteristic of the Attic Greeks, had by the time of Pythagoras given place to a well defined system of notation and computation by means of symbols. These consisted of the letters of the regular Greek alphabet, with three additions from an older alphabet, which were used to make up a decimal system of notation.* Although operations with these symbols were cumbrous and complicated, they were possible and the ordinary operations of addition, subtraction, multiplication, and division were quite fully developed.† They had made some progress in the theory of numbers which was denoted by the term $\alpha\rho\iota\theta\mu\eta\tau\iota\kappa\eta'$ as contrasted with $\lambda\omicron\gamma\iota\sigma\tau\iota\kappa\eta'$ the art of calculation. Their knowledge of Algebra, as an abstract science, was almost nothing, until after the time of Christ, although they had a slight conception of it, perhaps, from the Egyptians who were familiar with simple equations long before the time of Pythagoras.‡ But the Greeks had made unusual progress in Geometry, and of the problems of Algebra which can easily be given a geometric interpretation and solved by methods essentially geometrical, they had a very considerable knowledge which we shall investigate a little more in detail. Accordingly we shall take up the Greek methods of solving quadratic equations under two distinct heads, geometrical or constructive methods, and methods purely algebraic.

GEOMETRIC METHODS.

Pure Quadratics. 1. *Square Root.* Although not formulated by the Greeks in this way we may properly look upon their work in finding square roots of numbers as the solution of the pure quadratic $x^2 = a$, and accordingly as the first work which they did in the solution of quadratic equations.

The Pythagoreans are credited with the first knowledge of the fact that in a square

$$\text{diagonal} : \text{side} :: 1/\sqrt{2} : 1,$$

and of the method by which it was established, but it was kept a profound secret from their contemporaries.§ Although there was this geometrical

* See Cantor, pp. 110-119; Cajori, p. 64; Heath, *Archimedes*, p. lxi; Nesselmann, p. 78, et seq.

† Numerous problems given in Gow; Heath, *Archimedes*; Tannery; Hankel.

‡ See Cantor, pp. 37, 38, for numerous examples of equations from the Ahmes papyrus.

§ It is interesting to note the story that the Pythagorean who first divulged this knowledge of the irrationals perished in a shipwreck as an evidence of the displeasure of the gods. A similar story is told of the discloser of the knowledge of the dodecahedron. See Allman, pp. 25, 47.

equivalent for $\sqrt{2}$, they also recognized its irrationality but had no numerical expression for it, because they did not recognize irrationals as members of their number system; when they occurred they were rejected in the same way that imaginaries were in later centuries.* Lines were the indispensable symbols for irrationals because they avoided the necessity for numerical expression. But in the further development of Geometry, especially about the time of Euclid, it became necessary to obtain approximate values for some square roots, and the results and methods by which they were obtained are worth consideration.

Archimedes, perhaps the greatest mathematician of antiquity, is the first to give us any number of them. But unfortunately he gives only results and no suggestion of the method by which he reached them, nor is there any example of the actual complete calculation of a root extant by anyone before the time of Christ. Archimedes in his *Circuli Dimensio*† in dealing with the problem of finding an approximate value for the ratio of circumference to diameter says that $\sqrt{3}$ lies between $\frac{1}{7}\frac{3}{80}\frac{1}{1}$ and $\frac{2}{15}\frac{5}{3}$, that is, between 1.7320513 and 1.7320327. Now $\sqrt{3}$ correct to seven decimal places is 1.7320508, so that his approximation is remarkably accurate, his upper limit being exactly equal to six places. Many ingenious theories have been proposed to explain how Archimedes could secure this result, among them a method by continued fractions, one by an approximation in the form of a series of fractions, Theon's method of sexagesimal fractions based on Euclid (illustrated later, see p. 7), and others.‡ But we have no positive knowledge as to how he secured this result. If it were for $\sqrt{3}$ alone we might accept the theory that his only method consisted in guessing at the result and gradually making successive guesses more nearly accurate. But this seems highly improbable in view of the following results which occur in the same connection in Archimedes.§ (I have added the correct results to two places in the third column.)

| | |
|--|----------|
| 3013 $\frac{3}{4}$ is greater than $\sqrt{9082321}$ | 3013.69 |
| 1838 $\frac{9}{11}$ is greater than $\sqrt{3380929}$ | 1838.73 |
| 1009 $\frac{1}{6}$ is greater than $\sqrt{1018405}$ | 1009.16 |
| 2017 $\frac{1}{4}$ is greater than $\sqrt{4069284 \frac{1}{36}}$ | 2017.247 |
| 591 $\frac{1}{8}$ is less than $\sqrt{349450}$ | 591.14 |
| 1172 $\frac{1}{8}$ is less than $\sqrt{1373943 \frac{33}{64}}$ | 1172.15 |
| 2339 $\frac{1}{4}$ is less than $\sqrt{5472132 \frac{1}{16}}$ | 2339.26 |

* See Cantor, p. 170.

† Prop. 3. Heiberg, *Archimedes*, p. 262 et seq. Heath, p. 93.

‡ Heath refers to Gunther, *Die quadratischen Irrationalitäten der Alten und deren Entwicklungsmethoden*, Leipzig, 1882, as an exhaustive paper discussing in detail all the hypotheses offered up to 1882 for finding $\sqrt{3}$, including those of Zeuthen, Tannery, DeLagny, Heilermann, and Rodet.

§ Prop. 3. Heiberg, *Archimedes*, p. 262 et seq. These values occur in finding the value of π by the successive steps necessary in inscribing in a circle a regular polygon of 96 sides.

These are very close approximations and those are especially interesting where the number whose square root is sought is itself fractional. It is quite clear from these examples that Archimedes must have had a definite method to secure such accurate results with these large and varied numbers.* It is worth noting in passing that the value of π which he secured by means of these numbers was between $3\frac{1}{7}$ and $3\frac{10}{71}$, that is, between 3.1429 and 3.1408.†

A century and a quarter later Heron used as a formula for computing square root, $\sqrt{a^2 \pm b}$ is approximately $a \pm \frac{b}{2a}$.‡ He gives as roots determined thus:

$$\sqrt{50} \text{ is } 7 + \frac{1}{14}, \text{ i. e. } 7.071.$$

$$\sqrt{63} \text{ is } 8 - \frac{1}{16}, \text{ i. e. } 7.937.$$

$$\sqrt{75} \text{ is } 8 + \frac{11}{16}, \text{ i. e. } 8.687.$$

The first two are correct to three places, but the third where the “ b ” is not unity is correct only to the first place, the true value being 8.660. Heron’s method is probably based on Euclid, being an adaptation of the process given in the next paragraph. He also used for greater accuracy $a \pm \frac{b}{2a} > \sqrt{a^2 \pm b} > a \pm \frac{b}{2a \pm 1}$, by which $\sqrt{75}$ would be between 8.687 and 8.647. From the fact that Archimedes’ results all appear as “greater than” or “less than” the real value it would seem that he might have used some similar formula; but of course Heron’s is not accurate enough for the results given. Heron also gives $7/5$ for $\sqrt{2}$ (correct value 1.414+), and $26/15$ for $\sqrt{3}$, i. e. 1.733 (correct value 1.732+). In his *Stereometrica* is the first known attempt to express the square root of a negative number. Without method or comment $\sqrt{81-144}$ is stated to be 8 less than $1/16$!

The last method of finding square root we shall give is found in the works of Theon,§ very late, in the fourth century after Christ. It makes use of the sexagesimal system of angles of the Babylonians but is not essentially different from our present method. Although based on Euclid, Theon’s language would indicate that the method itself is comparatively new. As showing the antiquity of our method in its essence and illustrating the pro-

* Heath says “There is no doubt that in obtaining the integral portion of the square root” he used the method of Theon, illustrated in the next paragraph. Gow: “It is clear that Archimedes did not use Theon’s method!” And there are various shades of opinion between these extremes. See previous references to Gunther. Also Cantor, p. 301 et seq. Heath, *Archimedes*, p. lxxx, gives in detail Hultsch’s theory of “tentative assumptions” ingeniously worked out for many of these results, with several circumstances leading to its probability. See also Nesselmann, p. 108 et seq.

† Heiberg, *Archimedes*, Vol. I, p. 270.

‡ Fink, p. 70.

§ In his commentary on Ptolemy’s *Almagest* (A. D. 125).

cedure of at least the later Greeks we give the following example paraphrased from Theon.*

'I ought to mention how we extract the approximate root of a quadratic which has only one irrational root. We learn the process from Euclid, II, 4, where it is stated 'if a straight line be divided at any point, the square of the whole line is equal to the square of both the segments together with twice the rectangle contained by the segments.' So with a number like 144, which has a rational root, as the line $a\beta$ (see Fig. I) we take a lesser square say 100, of which the root is 10, as $a\gamma$. We multiply 10 by 2 because there are two rectangles, and divide 44 by 20. The remainder 4 is the square of $\beta\gamma$ which must be 2. Let us try the number 4500 (see Fig. II), of which the root is $67^{\circ} 4' 55''$. Take a square $a\beta\gamma\delta$ containing 4500 degrees ($\mu\omicron\iota\rho\alpha\iota$). The nearest square number is 4489, of which the side (root) is 67° . Take $a\eta=67^{\circ}$ and $a\epsilon\xi\eta$ the square of $a\eta$. The remaining gnomon $\beta\xi\delta$ contains 11° or $660'$. Now divide $660'$ by $2 a\eta$, i. e. by 134. The quotient is $4'$. Take $\epsilon\theta$, $\eta\kappa=4'$ and complete the rectangles $\theta\xi$, $\xi\kappa$. Both these rectangles contain $536'$ ($268'$ each). There remains $124'=7440''$. From this we must subtract the square $\xi\lambda$ containing $16''$. The remaining gnomon $\beta\lambda\delta$ contains $7424''$. Divide this by $2 a\kappa=134^{\circ} 8'$. The quotient is $55''$. The remainder is $46'' 40'''$, which is the square of $\lambda\gamma$, of which the side is $55''$ nearly enough'.

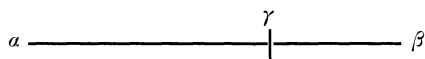


Fig. I.

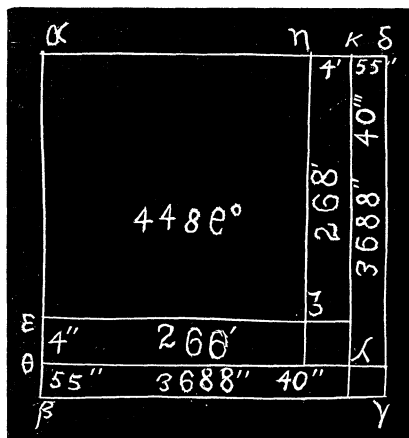


Fig. II.

Theon also gives another example with a figure for $\sqrt{2^{\circ} 28'}$ which he finds to be $1^{\circ} 34' 15''$. This is correct to $.8'$. The procedure is the same as above. Theon concludes: "When we seek a square root we first take the root of the nearest square number. We then double this and divide with it the remainder reduced to minutes, and subtract the square of the quotient. Then we reduce the remainder to seconds, and divide by twice the degrees and

* Gow, p. 55. See also Cantor, pp. 460-461; Heath, *Archimedes*, p. lxxvi.

minutes (of the whole quotient). We thus obtain nearly the root of the quadratic." Except for the sexagesimal notation this does not sound very different from the rule in our elementary texts today.

By this process Ptolemy gives $\sqrt{3} = \frac{103}{60} + \frac{55}{60^2} + \frac{23}{60^3}$, which in our notation is 1.7320509 and so correct to 6 places.

2. *Constructive methods.* If the pure quadratic is considered in the form " $x^2=ab$ " it means in geometrical language to find a square equivalent to a given rectangle, and this was solved by Euclid and perhaps his predecessors, in the two propositions, "to construct a square equal to a given rectilinear figure"* and "to two given straight lines to find a mean proportional"† being the same methods given in our elementary geometries today.

If the quadratic is in the form " $x^2=pa^2$ " it becomes geometrically the problem of the multiplication of the square, which is solvable by the Pythagorean theorem.‡ In its special form " $x^2=2a^2$ " it becomes the problem of the duplication of the square, which we have already noted was first known by the Pythagoreans. Perhaps its successful solution suggested the more famous problem of antiquity, the duplication of the cube.

Affected Quadratics. By geometrical methods the Greeks were able to solve any equation of the type $x^2+px+q=0$ where a real solution was possible, although they did not consider it thus generally. Rather they considered quadratics under three forms,

$$(a) \quad x^2+px=q$$

$$(b) \quad x^2+q=px$$

$$(c) \quad x^2=px+q$$

due to the fact that the treatment of them was geometric. Their solution consisted principally in applying theorems on areas, one of the most powerful methods on which Greek geometry relied, or on proportion, in which they were also well versed.

The procedure may be shown by a typical example, of the first type,

$$x^2+ax=b. \S$$

Expressed geometrically, this would be "To the segment $AB=a$ (see Fig. III) apply the rectangle DH , of known area, b , in such a way that CH shall be a square." The figure shows that for $CK=a/2$

* Euclid, II, 14. Heiberg, *Euclid*, Vol. I, p. 160.

† Euclid, VI, 13. Heiberg, *Euclid*, Vol. II, p. 110.

‡ Euclid, I, 47. Heiberg, *Euclid*, Vol. I, p. 110.

§ Fink, p. 79. Zeuthen, p. 36 et seq., gives several more.

$$FH=x^2+2(a/2)x+(a/2)^2=b^2+(a/2)^2.$$

But by the Pythagorean theorem

$$\begin{aligned} b^2+(a/2)^2 &= c^2, \\ \text{whence } EH=c &= a/2+x \\ \text{and } x=c-a/2 &= BC. \end{aligned}$$

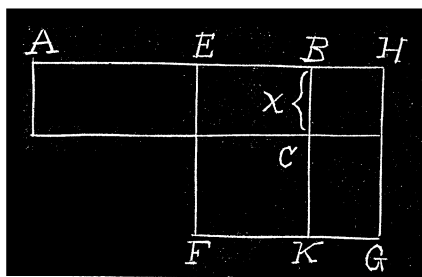


Fig. III.

In the same way Euclid solves all problems of this form and says that b must be greater than $a/2$ in $\sqrt{b^2-(a/2)^2}$ in order to give a solution. Thus imaginary roots are excluded.

The equation just solved is simply another way of expressing $x(x+p)=q$, which stated in Euclidean language would be "To produce a given line p to length $p+x$ so that the rectangle between the whole line so produced and the part produced, *i. e.* $x(x+p)$ shall be equal to a given figure q ." Similarly, $x^2=a(a-x)$ of the third type is Euclid's proposition "To cut a given straight line so that the rectangle contained by the whole and one of the segments is equal to the square on the remaining segment."* And Euclid finds $x=\sqrt{a^2+(a/2)^2}-a/2$.† It is certain that this particular problem, and probably others similar, were solved much earlier than Euclid, even by the Pythagoreans.‡

Their general method of solution by areas may be stated as follows. The problem was to apply to a given line a rectangle or more generally a parallelogram so that it would either contain a given area, or be greater or less than the given area by a constant. For these three conditions there arose probably even among the Pythagoreans the names *παρὰ βολή* (parallel to, application, equal), *ὑπερβολή* (excess), and *ἐλλείψις* (falling short).§ After the time of Archimedes, however, Apollonius took these names for the conic sections because they were appropriate to the distinctive properties of the parabola, hyperbola, and ellipse as he defined them.

The solution of affected quadratics by proportionality of lines was more general, but a later development probably not extensively used much before Euclid, while we have seen that the one by areas antedated him two centuries or more. In his sixth book Euclid gives by proportion the equivalent of the solution of the general equation $ax^2 \pm \frac{b}{c}x = S$ subject to the condition for real roots.||

* Euclid II, 11. Heiberg, *Euclid*, Vol. 1, p. 152.

† Cantor, pp. 249-250.

‡ Heath, *Euclid*, Vol. I, note to II, 5, p. 384, and II, 11, p. 403.

§ For antiquity of these terms see Proclus' note to Euclid I, 44, in Allman, p. 24, and Heath, *Euclid*, Vol. I, p. 343, "These things (*i. e.* method of areas) are ancient and are the Discoveries of the Muse of the Pythagoreans."

|| VI, 27, 28, 29, 30. Heiberg, *Euclid*, Vol. II, pp. 158-172. Discussed in their algebraic bearing in Heath, *Euclid*, Vol. 2, p. 260 et seq. Cantor, p. 252.

Apollonius did the same thing by means of the conics, especially with the aid of the equation $y^2 = px \pm \frac{p}{a}x^2$, the equation which he uses to express the fundamental property of a central conic.*

We conclude accordingly that the Greeks before the beginning of the Christian era were able to solve any equation of the second degree, having two essentially different coefficients, *geometrically* for real positive roots.

ALGEBRAIC METHODS.

Beginnings. A great step forward in mathematical reasoning was made when the Greeks were enabled to divorce their Algebra from their Geometry and reason abstractly, without necessary connection with the concrete concepts of geometry. Only then did they commence to develop a real science of algebra as a distinct subject. As a separate science among the Greeks it seems to have had its beginnings at least as early as the second century before the Christian era, but it did not reach a comparatively consistent development until two or three centuries later under Diophantus, by far the most important name in the consideration of Greek Algebra.

Arabian authorities state that Hipparchus wrote a treatise on the solution of quadratic equations, but no traces of it exist today, and we have no way of knowing whether it was any advance over the geometrical methods we have been considering.†

Heron a little later was the first to adopt the algebraic method, demonstrated the first ten or twelve propositions of Euclid, Book II, by means of lines only, without reference to areas, and dealt with them analytically as representations of pure numbers. He seems to have solved the affected quadratic equation $ax^2 + bx = c$ by completing the square, but the evidence is not conclusive.‡

Diophantus. There is much dispute as to the date of Diophantus but probably it is safe to say he flourished in the last half of the third century of our era.§

La Grange speaks of him as “l’inventeur de l’Algebre”|| while Tannery is unwilling to credit him with being anything more than a compiler. Nesselmann takes the intermediate view that the greater part of his propositions and ingenious methods are his own.¶ It seems fair to say that he did for Algebra what Euclid did for Geometry, organizing previous knowledge and adding much due to his own genius. His principal treatise is on “Arithmetic” (Ἀριθμητικῶν βιβλίον ἑκ) in six books of which a portion is

* Heath, *Apollonius*, p. cx.

† Cantor, p. 346.

‡ See Cantor, p. 376 et seq.

§ Fully Discussed in Heath, *Diophantus*, pp. 3-17.

|| La Grange, Vol. 7, p. 219.

¶ Nesselmann, p. 477. For full discussion of question of Diophantus' originality see Heath, *Diophantus*, Chap. VII, Cantor, p. 427 et seq.

lost; it is the first treatise on Algebra extant.* He was the first to use symbols for operations and for an unknown quantity, although he does not claim this symbolism is original. The unknown quantity was called 'ο 'αριθμός and its symbol was s' or $s^{0'}$, plural ss or $ss^{0'}$. The square of the unknown was $\delta\upsilon\nu\alpha\mu\iota s$ and its symbol δ^u . Symbols were used for as high as the sixth power of the unknown.† One of his equations looks like this

$$\begin{array}{ccccccccccc} \widehat{\kappa^u} & \beta & \delta^u & \bar{a} & 'i's\eta & ss^{0'is} & 'o & \phi & \mu^{0'} & \bar{i}\beta \\ x^3 & 2 & x^2 & 1 & \text{equals} & x & 4 & \text{minus} & \text{units} & 12 \end{array}$$

i. e. $2x^3 + x^2 = 4x - 12$. The part of Diophantus' work which deals with the solution of determinate quadratics is lost and we have little information as to its contents. Most of his extant work deals with the solution of indeterminate equations, including quadratics. His work shows deficiency in generalization but unusual ingenuity in the manipulation of processes, especially necessary since he never uses more than one symbol for the unknown, and other unknowns must be expressed in terms of it.

A single solution is sufficient for any set of conditions. Even when they lead to a quadratic he gives but one root, showing his ignorance of their true nature.‡ He refuses negative or imaginary roots as "impossible" or "absurd," but distinguishes between rational and irrational roots seeking only the former. His fundamental basis of classification of equations is not according to degree but *number of terms* when reduced to simplest form, *i. e.*, $ax^n = c$ is a "simple" equation whatever the power of x , while $ax^2 + bx = c$ is a "mixed" equation with two terms. In what follows we take that part of his work, for the most part, which deals with what we know as quadratic equations, although, as pointed out, this distinction is not always made by him.

Pure Quadratics. In Definition 11 of the First Book§ he gives his method for solving problems leading to pure quadratics. "If a problem leads to an equation containing the same powers of the unknown ($\epsilon'i'\delta\eta\tau\alpha\upsilon\tau\alpha$) on both sides but not with the same coefficients ($\mu\eta'$ 'ομοπληγ'οη) you must deduct like from like until only equal terms remain. But when on one side or both some terms are negative, ($\epsilon'\nu'$ 'ελλε'ιψεσι) you must add the negative terms to both sides till all the terms are positive ($\epsilon'\nu\nu\pi\alpha'\rho\chi\omicron\nu\tau\alpha$) and then deduct as before stated. If he comes to an equation such as $x^2 = ax$ he merely divides by the factor $x=0$ which he does not regard as a solution, *e. g.*,

* Diophantus announces 13 books. The seven existing mss. are in six books except one, which is in seven but has the same material. Probably we have the bulk of the original 13 books, with some sections lost (notably those on solution of determinate quadratics) and the remainder rearranged by later editors. See Nesselmann, p. 265 et seq.; Gow, p. 101-102; Heath, *Diophantus*, Chap. II.

† For symbolism see Nesselmann, pp. 294-296; Heath, *Diophantus*, Chap. IV; Cantor, p. 439.

‡ Hankel (p. 162) thinks this is a relic of the old geometric practice. Gow suggests that it was because Algebra was the invention of practical men who needed only the one solution.

§ Tannery, Vol. I, p. 14.

“ $20x=10x^2$ whence $x=2$.”*

Affected Quadratics. In Definition 11, Diophantus promises to give his method for solving “mixed” or affected quadratics. He continues: “We must contrive always, if possible, to reduce our equations so that they may contain one single term equated to one other. But afterward we will explain to you also how, when two terms are left equal to a third, such a question is solved.” That is, “reduce to the form $x^2=a$ if possible. Later we will give a method for the complete quadratic $x^2+ax=b$.” But this method, if ever written, is in the part which is lost. Without the method of solution he states numerous results, e. g., “ $84x^2+7x=7$, whence x is found equal to $\frac{1}{4}$ ”†; “ $630x^2+73x=6$, whence the root is rational,”‡ and many similar ones. When a root is irrational he sometimes gives an approximation to it.§ It is unfortunate that he nowhere gives an example of the complete solution of a single equation, but the form of his solution of equations of this type

$$ax^2+bx+c=0$$

is usually

$$x=\frac{(1/2)b\pm\sqrt{(1/4)b^2-ac}}{a}.$$

This is exactly the form it would have, had he completed the square after first multiplying the equation through by the coefficient of the x^2 term. The consensus of opinion seems to be that this is the way in which he arrived at his results.|| From the variety of equations which arise in Diophantus’ work we must believe that he had a general method of solution for all *determinate* quadratics, sufficient to determine one real root if such existed.

No sufficient idea can be given of the general methods he used in resolving *indeterminate* equations into solvable forms¶, but two examples will be given to show his ingenuity and some of his methods of attack.

A common method of attack is that of *tentative assumptions*, in which a value is shrewdly assumed and then altered as occasion arises until it fits the conditions of the problem. It frequently involves the use of his single symbol for the unknown quantity in different senses. For example, in book four° the problem is “to find three numbers such that their sum is a square

* I. 31; Tannery, Vol. I, p. 68, and many others in first book.

† VI. 6. Tannery, p. 404.

‡ VI. 8. Tannery, p. 408.

§ E. g. V, 30. Tannery, p. 384.

¶ See Nesselmann, p. 319; Cantor, p. 443 et seq; Heath, *Diophantus*, pp. 90-92.

|| Hankel (pp. 164-165) says it is difficult after solving 100 of Diophantus’ problems to have any idea how to attack the 101st. Nesselmann gives an unsatisfactory classification and discussion of methods of solution (pp. 315-328) and says himself that a complete discussion would mean copying Diophantus’ entire work. Heath disagrees with both and devotes Chap. V of his book on Diophantus to a fairly satisfactory classification and discussion.

° IV, 17. Tannery, pp. 222-225. Gow, p. 117.

and the square of any number plus the following one is a square." He assumes the three numbers first as $(x-1)$ $(4x)$ $(8x+1)$ where $(x-1)^2 + 4x$ and $(4x)^2 + (8x+1)$ are both square numbers. Two of the conditions are thus satisfied, but the third is that the sum of the numbers, $13x$, must also be a square number. He says: "Take $13x = x^2$ with some square coefficient e. g. $13x = 169x^2$. Then $x = 13x^2$." A new use of " x " is thus introduced and $13x^2$ is substituted for the original x , the numbers now being, $13x^2 + 1$, $52x^2$ and $104x^2 - 1$. A fourth condition remains, viz, that

$$(104x^2 + 1)^2 + (13x^2 - 1)$$

shall be a square. Diophantus then takes this expression equal to

$$x^2(104x^2 + 1)^2$$

and finds $x = \frac{111}{104}$, and substitutes this value, finally getting his three numbers

$$\frac{170989}{10816}, \frac{540692}{10816}, \frac{1270568}{10816}.$$

Of course this has all been expressed in modern notation. His last number, for example, is expressed

$$\frac{a}{\rho\kappa\zeta} : \frac{\omega\iota\varsigma}{\phi\zeta\eta}.$$

Another frequent artifice which he uses is a method of *limits*. If he wishes to find a square number, between 10 and 11 for example, he multiplies these by successive squares until a square number lies between the product; thus between 40 and 44 or 90 and 99 no (rational) square lies, but between 160 and 176 is 169; hence $x^2 = \frac{169}{160}$ will lie between the proposed limits. This is made use of in the problem: "To divide 1 into two parts such that if 3 be added to the first part and 5 to the other, the product of the two sums shall be a square."* If one part be $x-3$ the other is $4-x$. Then $x(9-x)$ must be a square. Suppose it equals $4x^2$, then $x = \frac{9}{5}$. But this will not suit the original assumption since x must be greater than 3 and less than 4. Now 5 is $4+1$. Hence what is wanted is to find a number $y^2 + 1$, such that $\frac{9}{y^2 + 1}$ is greater than 3 and less than 4. For such a purpose y^2 must be less than 2 and greater than $1\frac{1}{4}$. "I resolve these expressions into square

* IV, 31; Tannery, pp. 278-283; Gow, p. 119. For this method see also Nesselmann, p. 318; Heath, *Diophantus*, pp. 115-120.

fractions" says Diophantus, selecting $\frac{1\frac{3}{4}}{6\frac{3}{4}}$ and $\frac{8\frac{0}{6}}{6\frac{3}{4}}$ between which lies the square $\frac{1\frac{0}{6}}{6\frac{3}{4}}$ or $\frac{2\frac{5}{1}}{6}$. He then puts $x(9-x) = \frac{25x^2}{16}$ instead of $4x^2$, whence $x = \frac{1\frac{4}{4}}{4\frac{1}{1}}$, and the two numbers are $\frac{3\frac{1}{4}}{4\frac{1}{1}}$ and $\frac{2\frac{0}{4}}{4\frac{1}{1}}$.

In general we may say that the type of indeterminate quadratic equations which he considers fully are limited to the case where one or two functions of the unknown in the form $Ax^2 + Bx + C$ must be a rational square, and are only fully treated in cases where the "C" vanishes. Otherwise his solution depends upon the particular equation involved, and consists of ingenious transformations or assumptions of the unknown.† The most characteristic feature of his work is the extraordinary artifices which he employs.

Such then are the principal methods known to the ancient Greeks for solving quadratic equations. In the classic period they had a comparatively complete knowledge of methods of solution from the purely geometric standpoint, practically none algebraically. Even in later times their knowledge of the algebraic solution was faulty and cramped, showing no true conception of the real nature of the quadratic, only familiarity with ingenious methods of manipulating particular equations. Considering the exceptionally high attainments of the Greeks in geometry as well as other branches of learning and culture, it seems a little strange that they did not make greater advances in algebra. This was reserved for the Hindoos some centuries later. Nevertheless it is instructive to trace out some of their attainments in this particular line in order to note their limitations as well as their achievements.

† Heath, *Diophantus*, p. 115.